

Question 4

Consider the “monkey saddle”

$$f(x, y) = x^3 - 3xy^2.$$

Tasks

1. Find all critical points.
2. Show precisely why the second derivative test is inconclusive there.
3. Classify the point using both simple paths and the polar form of the function.

Question 4 – Solution

Strategy. When the Hessian degenerates, return to the definition and compare the function along paths through the critical point.

Step 1: Critical point

$$f_x = 3x^2 - 3y^2, \quad f_y = -6xy.$$

The second equation requires $x = 0$ or $y = 0$. In either case the first equation then forces the other coordinate to be zero. Thus the only critical point is $(0, 0)$.

Step 2: Failed Hessian test

$$f_{xx} = 6x, \quad f_{yy} = -6x, \quad f_{xy} = -6y.$$

All three second derivatives vanish at the origin, so

$$D(0, 0) = 0.$$

The test gives no classification.

Step 3: Direct classification

Along $y = 0$,

$$f(x, 0) = x^3,$$

which takes both signs arbitrarily close to zero. Hence the origin is a saddle.

For a structural check, let $x = r \cos \theta$ and $y = r \sin \theta$. Then

$$f = r^3 (\cos^3 \theta - 3 \cos \theta \sin^2 \theta) = r^3 \cos(3\theta).$$

The factor $\cos(3\theta)$ alternates sign in six angular sectors. Therefore

