

Question 3

Analyze the origin for

$$f(x, y) = x^2 + 4xy + y^2.$$

Tasks

1. Verify that the origin is the only critical point.
2. Classify it using the second derivative test.
3. Give two explicit paths that make the saddle geometry visible.

Question 3 – Solution

Strategy. The quadratic form itself records the local geometry; the Hessian test and carefully chosen diagonal paths should agree.

Step 1: Critical point

$$f_x = 2x + 4y, \quad f_y = 4x + 2y.$$

Solving $x + 2y = 0$ and $2x + y = 0$ gives $x = y = 0$. Therefore the origin is the unique critical point.

Step 2: Hessian test

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 4.$$

Hence

$$D = (2)(2) - 4^2 = -12 < 0.$$

Therefore

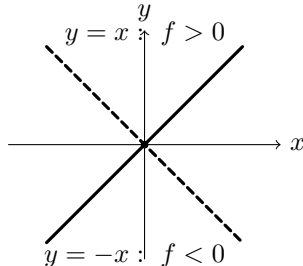
$(0, 0)$ is a saddle point.

Step 3: Path verification Along $y = x$,

$$f(x, x) = x^2 + 4x^2 + x^2 = 6x^2 > 0$$

for $x \neq 0$. Along $y = -x$,

$$f(x, -x) = x^2 - 4x^2 + x^2 = -2x^2 < 0.$$



Both paths approach the origin, whose value is zero, so arbitrarily nearby values occur on both sides of the critical value.