

Question 2

For

$$f(x, y) = x^3 + y^3 - 3xy,$$

find and classify every critical point.

Tasks

1. Solve the nonlinear critical-point equations without losing solutions.
2. Apply the second derivative test at each point.
3. Verify the saddle classification by displaying nearby values of opposite signs relative to the critical value.

Question 2 – Solution

Strategy. Use the coupled equations to reduce the possible coordinates, then apply the Hessian determinant separately at each solution.

Step 1: Critical points

$$f_x = 3x^2 - 3y, \quad f_y = 3y^2 - 3x.$$

Thus $y = x^2$ and $x = y^2$. Substitution gives

$$x = x^4, \quad x(x^3 - 1) = 0.$$

The real possibilities are $x = 0$ and $x = 1$, giving

$$\boxed{(0, 0) \quad \text{and} \quad (1, 1)}.$$

Step 2: Hessian test

$$f_{xx} = 6x, \quad f_{yy} = 6y, \quad f_{xy} = -3,$$

so $D = 36xy - 9$. At $(0, 0)$, $D = -9 < 0$, hence it is a saddle point. At $(1, 1)$, $D = 27 > 0$ and $f_{xx} = 6 > 0$, so it is a strict relative minimum with

$$f(1, 1) = \boxed{-1}.$$

Step 3: Saddle verification At the origin the critical value is 0. Along $y = 0$,

$$f(x, 0) = x^3,$$

which is positive for small $x > 0$ and negative for small $x < 0$. Thus every neighborhood contains values above and below 0, independently verifying the saddle classification.