

Question 1

For

$$f(x, y) = x^2 + 2y^2 - 4x + 8y + 1,$$

analyze all relative extrema.

Tasks

1. Find every critical point.
2. Classify each point with the second derivative test.
3. Complete the square to verify the classification and find the extremal value.

Question 1 – Solution

Strategy. Solve the first-derivative equations, classify with the Hessian determinant, and then use an exact algebraic form as an independent check.

Step 1: Critical point

$$f_x = 2x - 4, \quad f_y = 4y + 8.$$

Both vanish only when

$$\boxed{(x, y) = (2, -2)}.$$

Step 2: Second derivative test

$$f_{xx} = 2, \quad f_{yy} = 4, \quad f_{xy} = 0.$$

The Hessian determinant is

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 8 > 0.$$

Since $f_{xx} > 0$, the point is a strict relative minimum. Its value is

$$f(2, -2) = \boxed{-11}.$$

Step 3: Algebraic verification Completing squares gives

$$f(x, y) = (x - 2)^2 + 2(y + 2)^2 - 11.$$

Both squared terms are nonnegative and vanish simultaneously only at $(2, -2)$. This directly confirms the strict local minimum and rules out any other critical point or relative extremum.