

Question 10

On the ellipsoid

$$x^2 + 2y^2 + 3z^2 = 6,$$

find every point at which the normal line passes through the origin.

Tasks

1. Derive the condition relating a point to its gradient.
2. Prove that at most one coordinate of such a point can be nonzero.
3. List all points, tangent planes, and normal lines, and verify completeness.

Question 10 – Solution

Strategy. If the origin lies on the normal line at P , then the vector from P to the origin must be parallel to $\nabla F(P)$.

Step 1: Parallelism condition Write $P = (p, q, r)$. Since

$$\nabla F(P) = \langle 2p, 4q, 6r \rangle,$$

the origin lies on the normal line exactly when some t satisfies

$$\langle -p, -q, -r \rangle = t \langle 2p, 4q, 6r \rangle.$$

If $p \neq 0$, then $t = -1/2$; if $q \neq 0$, then $t = -1/4$; if $r \neq 0$, then $t = -1/6$. One value of t cannot meet two of these conditions, so at most one coordinate is nonzero.

Step 2: Points and planes Using the ellipsoid equation on each coordinate axis gives exactly six points:

$$(\pm\sqrt{6}, 0, 0), \quad (0, \pm\sqrt{3}, 0), \quad (0, 0, \pm\sqrt{2}).$$

Their tangent planes are, respectively,

$$\boxed{x = \pm\sqrt{6}}, \quad \boxed{y = \pm\sqrt{3}}, \quad \boxed{z = \pm\sqrt{2}}.$$

Step 3: Normal lines and verification The corresponding normal lines are

$$\boxed{\mathbf{r} = (\pm\sqrt{6}, 0, 0) + t \langle 1, 0, 0 \rangle},$$
$$\boxed{\mathbf{r} = (0, \pm\sqrt{3}, 0) + t \langle 0, 1, 0 \rangle}, \quad \boxed{\mathbf{r} = (0, 0, \pm\sqrt{2}) + t \langle 0, 0, 1 \rangle}.$$

These are the coordinate axes written through their stated points. Every one passes through the origin and is perpendicular to its listed coordinate plane. The parallelism argument rules out mixed-coordinate points, while the surface equation rules out the origin itself, so the list is complete.