

Question 9

Let S be a smooth level surface $F(x, y, z) = c$, and let P be a point of S with $\nabla F(P) \neq 0$.

Tasks

1. For any differentiable curve $\mathbf{r}(t)$ on S through P , prove that its tangent at P is perpendicular to $\nabla F(P)$.
2. Deduce the tangent-plane equation at $P = (x_0, y_0, z_0)$.
3. Deduce the normal-line equation and explain the role of the regularity condition.

Question 9 – Solution

Strategy. Differentiate the constant identity $F(\mathbf{r}(t)) = c$ and interpret the chain rule geometrically.

Step 1: Tangent vectors Suppose $\mathbf{r}(t_0) = P$ and the curve lies on S . Then

$$F(\mathbf{r}(t)) = c.$$

Differentiating at t_0 by the multivariable chain rule gives

$$0 = \left. \frac{d}{dt} F(\mathbf{r}(t)) \right|_{t=t_0} = \nabla F(P) \cdot \mathbf{r}'(t_0).$$

Thus every surface-tangent velocity is perpendicular to $\nabla F(P)$.

Step 2: Tangent plane Because $\nabla F(P)$ is nonzero, it supplies a valid normal. Therefore

$$\boxed{F_x(P)(x - x_0) + F_y(P)(y - y_0) + F_z(P)(z - z_0) = 0}.$$

Step 3: Normal line The line through P in the gradient direction is

$$\boxed{\mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle + t \nabla F(P)}.$$

The condition $\nabla F(P) \neq 0$ ensures a genuine normal direction and a two-dimensional tangent plane. At a singular point, the zero vector cannot determine either object; additional geometric analysis is required.