

**Question 6**

Consider the double cone

$$F(x, y, z) = x^2 + y^2 - z^2 = 0.$$

**Tasks**

1. Find the tangent plane and normal line at the regular point  $P = (1, 0, 1)$ .
2. Explain why the gradient formula fails to select a plane at the origin.
3. Use generator curves through the origin to prove that no single tangent plane contains all tangent directions there.

## Question 6 – Solution

**Strategy.** Contrast a regular point, where the gradient is nonzero, with the cone vertex, where many generator directions meet and the gradient vanishes.

**Step 1: The regular point**

$$\nabla F = \langle 2x, 2y, -2z \rangle, \quad \nabla F(1, 0, 1) = \langle 2, 0, -2 \rangle.$$

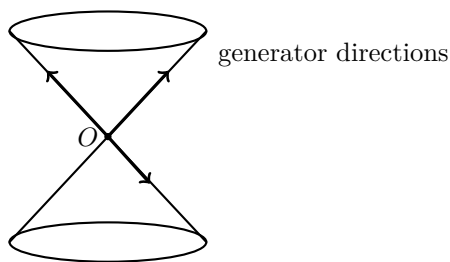
Thus

$$2(x - 1) - 2(z - 1) = 0, \quad \boxed{x - z = 0},$$

and a normal line is

$$\boxed{(x, y, z) = (1, 0, 1) + t(1, 0, -1)}.$$

**Step 2: The vertex** At  $O = (0, 0, 0)$ ,  $\nabla F(O) = \langle 0, 0, 0 \rangle$ , so the usual plane equation reduces to  $0 = 0$  and supplies no normal.



**Step 3: Generator argument** For every  $\theta$ , the curve

$$\mathbf{r}_\theta(s) = \langle s \cos \theta, s \sin \theta, s \rangle$$

lies on the cone and has tangent  $\langle \cos \theta, \sin \theta, 1 \rangle$  at  $O$ . The directions for  $\theta = 0, \pi$ , and  $\pi/2$  span all of  $\mathbb{R}^3$ : the first two yield the  $x$ - and  $z$ -directions by differences and sums, and the third then yields the  $y$ -direction. No two-dimensional plane can contain them all. Hence  $\boxed{\text{there is no unique tangent plane at the vertex}}.$