

Question 5

For the surface

$$xy + yz + zx = 3,$$

consider $P = (1, 1, 1)$.

Tasks

1. Find the normal line at P .
2. Find every point where that normal line intersects the sphere $x^2 + y^2 + z^2 = 12$.
3. Verify both the surface-normal direction and the sphere intersections.

Question 5 – Solution

Strategy. Parameterize the normal line using the gradient, then substitute that line into the sphere equation.

Step 1: Normal line Let $F = xy + yz + zx$. Then

$$\nabla F = \langle y + z, x + z, x + y \rangle, \quad \nabla F(P) = \langle 2, 2, 2 \rangle.$$

Using the simplified direction $\langle 1, 1, 1 \rangle$,

$$\boxed{(x, y, z) = (1 + t, 1 + t, 1 + t)}.$$

Step 2: Intersections with the sphere Substitution gives

$$3(1 + t)^2 = 12,$$

so $(1 + t)^2 = 4$ and

$$t = 1 \quad \text{or} \quad t = -3.$$

The intersection points are therefore

$$\boxed{(2, 2, 2) \quad \text{and} \quad (-2, -2, -2)}.$$

Step 3: Verification The base point satisfies $1 + 1 + 1 = 3$, and the line direction is parallel to $\nabla F(P)$, so the line is normal to the original surface. Finally,

$$2^2 + 2^2 + 2^2 = 12, \quad (-2)^2 + (-2)^2 + (-2)^2 = 12,$$

confirming both sphere intersections.