

Question 4

On the ellipsoid

$$x^2 + 2y^2 + z^2 = 9,$$

find every point whose tangent plane is parallel to

$$2x - 4y + 2z = 7.$$

Tasks

1. Determine all points of tangency.
2. Write the tangent plane at each point.
3. Give a normal line at each point and verify completeness.

Question 4 – Solution

Strategy. Parallel tangent planes have parallel normal vectors; impose that condition together with the surface equation.

Step 1: Parallel-gradient condition For $F = x^2 + 2y^2 + z^2$,

$$\nabla F = \langle 2x, 4y, 2z \rangle.$$

The given plane has normal parallel to $\langle 1, -2, 1 \rangle$. Set

$$\langle 2x, 4y, 2z \rangle = \lambda \langle 1, -2, 1 \rangle.$$

Then $x = \lambda/2$, $y = -\lambda/2$, and $z = \lambda/2$. Substitution into the ellipsoid gives $\lambda^2 = 9$.

Thus the two points are

$$P_+ = \left(\frac{3}{2}, -\frac{3}{2}, \frac{3}{2} \right), \quad P_- = -P_+.$$

Step 2: Tangent planes Using the simplified common normal,

$$x - 2y + z = 6 \quad \text{at } P_+,$$

$$x - 2y + z = -6 \quad \text{at } P_-.$$

Step 3: Normal lines

$$\mathbf{r} = P_+ + t \langle 1, -2, 1 \rangle, \quad \mathbf{r} = P_- + t \langle 1, -2, 1 \rangle.$$

The equations $\lambda = \pm 3$ exhaust the surface constraint, proving there are no additional points.