

Question 10

Consider the family

$$f_{a,b}(x, y) = ax^2 + bxy + y^2.$$

At $(1, 1)$, its tangent plane passes through $(2, 0, 5)$, and its linearization predicts the value 3.06 at $(1.02, 0.99)$.

Tasks

1. Determine the parameters a and b .
2. Write the resulting tangent plane at $(1, 1)$.
3. Use it to estimate $f(0.98, 1.03)$ and verify both original constraints.

Question 10 – Solution

Strategy. Express the value and slopes at $(1, 1)$ in terms of a, b , then turn the two tangent-plane conditions into a linear system.

Step 1: General linearization At $(1, 1)$,

$$f = a + b + 1, \quad f_x = 2a + b, \quad f_y = b + 2.$$

Therefore

$$L(x, y) = a + b + 1 + (2a + b)(x - 1) + (b + 2)(y - 1).$$

Passing through $(2, 0, 5)$ requires

$$3a + b - 1 = 5, \quad \text{so} \quad 3a + b = 6.$$

The prediction at $(1.02, 0.99)$ requires

$$1.04a + 1.01b + 0.98 = 3.06.$$

Solving the two equations gives

$$\boxed{a = 2, \quad b = 0}.$$

Step 2: Tangent plane Now $f(1, 1) = 3$, $f_x(1, 1) = 4$, and $f_y(1, 1) = 2$. Hence

$$\boxed{z = 3 + 4(x - 1) + 2(y - 1)}.$$

Step 3: New estimate and verification At $(0.98, 1.03)$,

$$f(0.98, 1.03) \approx 3 + 4(-0.02) + 2(0.03) = \boxed{2.98}.$$

Also $L(2, 0) = 3 + 4 - 2 = 5$, and $L(1.02, 0.99) = 3 + 0.08 - 0.02 = 3.06$, verifying both supplied constraints.