

Question 8

Suppose a function f is differentiable at (a, b) . An affine function has the form

$$A(x, y) = c + p(x - a) + q(y - b).$$

Tasks

1. Prove that if A is a linear approximation to f at (a, b) , then $c = f(a, b)$.
2. Use displacements along the coordinate axes to prove $p = f_x(a, b)$ and $q = f_y(a, b)$.
3. Conclude that the tangent-plane linearization is unique.

Question 8 – Solution

Strategy. Apply the defining small-error condition first at zero displacement and then on each coordinate axis.

Step 1: Constant term For A to approximate f at the base point, it must agree there. Since $A(a, b) = c$,

$$\boxed{c = f(a, b)}.$$

Equivalently, the remainder must vanish when the displacement is zero.

Step 2: The x coefficient The approximation condition along $(h, 0)$ is

$$f(a + h, b) = f(a, b) + ph + o(|h|).$$

Subtract $f(a, b)$, divide by $h \neq 0$, and let $h \rightarrow 0$. Since $o(|h|)/h \rightarrow 0$ from both sides,

$$p = \lim_{h \rightarrow 0} \frac{f(a + h, b) - f(a, b)}{h} = \boxed{f_x(a, b)}.$$

Step 3: The y coefficient The same argument along $(0, k)$ yields

$$q = \lim_{k \rightarrow 0} \frac{f(a, b + k) - f(a, b)}{k} = \boxed{f_y(a, b)}.$$

Conclusion Every affine first-order approximation must therefore be

$$\boxed{L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)}.$$

All three coefficients are forced, proving uniqueness.