

Question 6

Let

$$f(x, y) = e^{x+2y}.$$

Use the linearization at $(0, 0)$ to estimate $f(0.02, 0.01)$ and rigorously bound the absolute error using the one-variable Taylor theorem for e^t .

Tasks

1. Find the linearization and the estimate.
2. Obtain a uniform error bound for $|x| \leq 0.02$, $|y| \leq 0.02$.
3. Compare the actual error at $(0.02, 0.01)$ with that bound.

Question 6 – Solution

Strategy. Write $t = x + 2y$ so the two-variable approximation reduces to the first-order Taylor approximation of e^t at zero.

Step 1: Linearization At the origin, $f = 1$, $f_x = 1$, and $f_y = 2$. Therefore

$$\boxed{L(x, y) = 1 + x + 2y}.$$

At $(0.02, 0.01)$, $t = 0.04$, so

$$f(0.02, 0.01) \approx 1.04.$$

Step 2: Uniform bound Taylor's theorem gives

$$e^t = 1 + t + \frac{e^\xi}{2}t^2$$

for some ξ between 0 and t . In the given rectangle,

$$|t| = |x + 2y| \leq 0.06,$$

so $e^\xi \leq e^{0.06}$. Consequently,

$$\boxed{|f - L| \leq \frac{e^{0.06}}{2}(0.06)^2 < 0.001912}.$$

Step 3: Actual error

$$|e^{0.04} - 1.04| \approx \boxed{0.0008108},$$

which lies below the uniform bound. The bound is larger because it must cover the entire rectangle, not only the requested point.