

Question 5

Define

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Tasks

1. Compute both partial derivatives at the origin from their definitions.
2. Identify the candidate tangent plane suggested by those partial derivatives.
3. Test the linear-approximation remainder along $y = x^2$ and decide whether a tangent plane exists there in the differentiability sense.

Question 5 – Solution

Strategy. Existing partial derivatives only suggest a plane; differentiability requires the remainder to be negligible compared with the distance to the base point.

Step 1: Axis partial derivatives Along either coordinate axis, f is zero. Hence

$$f_x(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0) - 0}{h} = 0, \quad f_y(0,0) = \lim_{k \rightarrow 0} \frac{f(0,k) - 0}{k} = 0.$$

Step 2: Candidate plane The value and both proposed slopes are zero, so the formal candidate is

$$\boxed{z = 0}.$$

Step 3: Remainder test Along $(x, y) = (t, t^2)$ with $t \neq 0$,

$$f(t, t^2) = \frac{t^4}{t^4 + t^4} = \frac{1}{2}.$$

For the candidate linearization $L = 0$, the required quotient is

$$\frac{|f(t, t^2) - L(t, t^2)|}{\sqrt{t^2 + t^4}} = \frac{1/2}{|t|\sqrt{1 + t^2}} \rightarrow \infty.$$

Therefore the error is not $o(\sqrt{x^2 + y^2})$.

Conclusion The function is not even continuous at the origin along this curve, so it is not differentiable there. Thus the graph has $\boxed{\text{no tangent plane at the origin in the linear-approximation sense}}.$