

Question 9

Let $f(x, y, z) = x^2 + y^2 + z^2$. At $P = (1, 2, 2)$, restrict directions to be tangent to the plane $x + y + z = 5$.

Tasks

1. Find the unit tangent direction giving the greatest increase of f .
2. Find that maximum constrained rate.
3. Identify the exceptional case in which no preferred tangent direction exists.

Question 9 – Solution

Strategy. Project the gradient onto the plane's tangent subspace, then normalize the projection.

Step 1: Data $\nabla f(P) = \langle 2, 4, 4 \rangle$. A unit normal to the plane is

$$n = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle.$$

The tangential projection is

$$p = \nabla f - (\nabla f \cdot n)n = \langle 2, 4, 4 \rangle - \frac{10}{3} \langle 1, 1, 1 \rangle = \frac{1}{3} \langle -4, 2, 2 \rangle.$$

Step 2: Direction and rate Since $\|p\| = 2\sqrt{6}/3$,

$$\boxed{u_{\max} = \frac{1}{\sqrt{6}} \langle -2, 1, 1 \rangle}, \quad \boxed{D_{\max, \text{tangent}} f = \frac{2\sqrt{6}}{3}}.$$

Step 3: Verify $u_{\max} \cdot \langle 1, 1, 1 \rangle = 0$, so it is tangent, and $\nabla f \cdot u_{\max} = 2\sqrt{6}/3$. If the projected gradient were zero, every tangent directional derivative would be zero and no unique maximizing tangent direction would exist.