

Question 7

Define

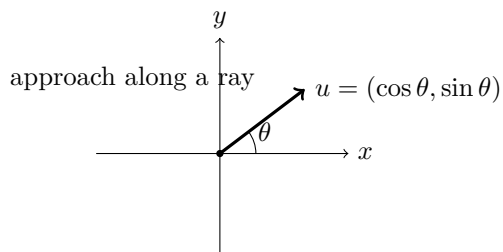
$$f(x, y) = \begin{cases} \frac{x^3}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Tasks

1. Compute $D_u f(0, 0)$ from the definition for every unit $u = \langle \cos \theta, \sin \theta \rangle$.
2. Compare with $\nabla f(0, 0) \cdot u$.
3. Explain what this proves about directional derivatives and differentiability.

Question 7 – Solution

Strategy. Approach the origin along the ray hu and keep the signed parameter h .



Step 1: Definition For $h \neq 0$,

$$f(h \cos \theta, h \sin \theta) = \frac{h^3 \cos^3 \theta}{h^2} = h \cos^3 \theta.$$

Therefore

$$D_u f(0, 0) = \lim_{h \rightarrow 0} \frac{h \cos^3 \theta}{h} = \cos^3 \theta.$$

Step 2: Gradient comparison Axis definitions give $f_x(0, 0) = 1$ and $f_y(0, 0) = 0$, so

$$\nabla f(0, 0) \cdot u = \cos \theta,$$

which differs from $\cos^3 \theta$ for most angles.

Conclusion Every directional derivative exists, but its dependence on u is not linear. Thus f is not differentiable at the origin, and the gradient formula is not valid there.