

Question 10

Let $z = f(x(t), y(t))$, where f has continuous second partials.

Tasks

1. Derive a formula for d^2z/dt^2 .
2. Organize it into curvature-of- f and acceleration-of-path terms.
3. Apply it to $f = x^2 + xy + y^2$, $x = \cos t$, $y = \sin t$ at $t = 0$.

Question 10 – Solution

Strategy. Differentiate $z' = f_x x' + f_y y'$ and apply the chain and product rules to every factor.

Step 1: General formula

$$z'' = f_{xx}(x')^2 + 2f_{xy}x'y' + f_{yy}(y')^2 + f_x x'' + f_y y''.$$

The first three terms measure the function's second-order response along the velocity; the last two measure the effect of path acceleration on the first partials.

Step 2: Data For $f = x^2 + xy + y^2$,

$$f_x = 2x + y, \quad f_y = x + 2y, \quad f_{xx} = 2, \quad f_{xy} = 1, \quad f_{yy} = 2.$$

At $t = 0$, $(x, y) = (1, 0)$, $(x', y') = (0, 1)$, and $(x'', y'') = (-1, 0)$.

Step 3: Evaluate

$$z'' = 2(0)^2 + 2(1)(0)(1) + 2(1)^2 + (2)(-1) + (1)(0) = \boxed{0}.$$

Directly, $z = \cos^2 t + \sin t \cos t + \sin^2 t = 1 + \frac{1}{2} \sin 2t$, whose second derivative is $-\sin 2t$, also zero at $t = 0$.