

Question 5

Let $u(x, y)$ be differentiable and set $x = r \cos \theta$, $y = r \sin \theta$.

Tasks

1. Derive formulas for u_r and u_θ .
2. Apply them to $u = x^2 - y^2$.
3. Verify against the polar form of u .

Question 5 – Solution

Strategy. Treat r and θ as independent inputs to the Cartesian coordinate map.

Step 1: General formulas

$$\boxed{u_r = u_x \cos \theta + u_y \sin \theta},$$

$$\boxed{u_\theta = -ru_x \sin \theta + ru_y \cos \theta}.$$

Step 2: Apply For $u = x^2 - y^2$, $u_x = 2x$, $u_y = -2y$. Substituting $x = r \cos \theta$, $y = r \sin \theta$ gives

$$u_r = 2r(\cos^2 \theta - \sin^2 \theta) = \boxed{2r \cos 2\theta},$$

$$u_\theta = -4r^2 \sin \theta \cos \theta = \boxed{-2r^2 \sin 2\theta}.$$

Step 3: Verify Directly, $u = r^2 \cos 2\theta$; its two ordinary partial derivatives match both formulas.