

Question 10

Define

$$f(x, y) = \begin{cases} \frac{x^3}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Tasks

1. Compute $f_x(0, 0)$ and $f_y(0, 0)$.
2. Test whether a differential $df = f_x dx + f_y dy$ gives a valid first-order approximation at the origin.
3. Identify a path that proves or disproves differentiability.

Question 10 – Solution

Strategy. Existing partial derivatives propose a linear map, but differentiability requires the remainder divided by distance to vanish along every path.

Step 1: Partial

$$f_x(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0)}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1,$$

$$f_y(0,0) = \lim_{k \rightarrow 0} \frac{f(0,k)}{k} = 0.$$

The candidate differential is therefore $L(dx, dy) = dx$.

Step 2: Remainder test Differentiability would require

$$\frac{f(x,y) - x}{\sqrt{x^2 + y^2}} \rightarrow 0.$$

Along $y = x$,

$$f(x,x) = \frac{x^3}{2x^2} = \frac{x}{2},$$

so

$$\frac{f(x,x) - x}{\sqrt{2x^2}} = \frac{-x/2}{\sqrt{2}|x|} = -\frac{\text{sgn}(x)}{2\sqrt{2}},$$

which does not tend to zero.

Conclusion Although both partials exist, f is not differentiable at the origin, so

$df = dx$ is not a valid total first-order approximation there.