

Question 10

Find all real constants a, b, c for which

$$u(x, y) = e^{ax+by} \cos(cx)$$

satisfies $u_{xx} + u_{yy} = 0$ everywhere.

Tasks

1. Compute the pure second partials.
2. Separate sine and cosine coefficients.
3. Solve all conditions, including $c = 0$.

Question 10 – Solution

Strategy. Equality for every x forces independent trigonometric coefficients to vanish.

Step 1: Differentiate

$$\begin{aligned}u_{xx} &= e^{ax+by}[(a^2 - c^2)\cos(cx) - 2ac\sin(cx)], \\u_{yy} &= b^2 e^{ax+by}\cos(cx).\end{aligned}$$

Thus the sum is

$$e^{ax+by}[(a^2 + b^2 - c^2)\cos(cx) - 2ac\sin(cx)].$$

Step 2: Nonzero c If $c \neq 0$, independence of sine and cosine gives

$$a^2 + b^2 - c^2 = 0, \quad ac = 0.$$

Thus $a = 0$ and $b = \pm c$.

Step 3: Edge case If $c = 0$, the condition is $a^2 + b^2 = 0$, hence $a = b = 0$. This is already included below:

$$(a, b, c) = (0, c, c) \text{ or } (0, -c, c), \quad c \in \mathbb{R}.$$

Substitution makes both coefficients vanish, verifying sufficiency.