

Question 6

Assume the mixed partials of f are continuous near (a, b) . Define

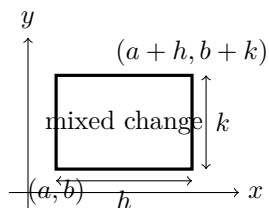
$$D(h, k) = \frac{f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)}{hk}.$$

Tasks

1. Interpret the numerator as a rectangular mixed change.
2. Apply the mean value theorem in each variable.
3. Find $\lim_{(h,k) \rightarrow (0,0)} D(h, k)$.

Question 6 – Solution

Strategy. Apply one-variable mean value arguments successively across the small rectangle.



Step 1: Difference structure Let

$$G(y) = f(a + h, y) - f(a, y).$$

The numerator is $G(b + k) - G(b)$.

Step 2: Mean value steps For some η between b and $b + k$,

$$G(b + k) - G(b) = kG'(\eta) = k[f_y(a + h, \eta) - f_y(a, \eta)].$$

For some ξ between a and $a + h$, the bracket equals $hf_{yx}(\xi, \eta)$. Hence

$$D(h, k) = f_{yx}(\xi, \eta).$$

Step 3: Limit As $(h, k) \rightarrow (0, 0)$, $(\xi, \eta) \rightarrow (a, b)$. Continuity and equality of mixed partials give

$$\boxed{\lim D(h, k) = f_{yx}(a, b) = f_{xy}(a, b)}.$$