

Question 5

A proposed Hessian is

$$H_f(x, y) = \begin{pmatrix} 6x - 2y & -2x + 4y \\ -2x + 4y & 4x + 6y \end{pmatrix}.$$

Tasks

1. Check its compatibility conditions.
2. Recover one possible $f(x, y)$.
3. Describe every function with this Hessian.

Question 5 – Solution

Strategy. Integrate f_{xx} and use the other entries to determine the unknown functions of integration.

Step 1: Compatibility The off-diagonal entries agree, and

$$\partial_y f_{xx} = -2 = \partial_x f_{xy}, \quad \partial_y f_{xy} = 4 = \partial_x f_{yy}.$$

Step 2: Reconstruct

$$f_x = 3x^2 - 2xy + A(y).$$

Since $f_{xy} = -2x + A'(y) = -2x + 4y$, $A(y) = 2y^2 + C$. Integrating in x ,

$$f = x^3 - x^2y + 2xy^2 + Cx + B(y).$$

The condition $f_{yy} = 4x + B''(y) = 4x + 6y$ gives $B = y^3 + Dy + E$.

Result

$$\boxed{f = x^3 - x^2y + 2xy^2 + y^3 + Cx + Dy + E}.$$

Conversely, direct differentiation shows every choice of C, D, E has exactly the proposed Hessian.