

Question 9

Define

$$u(x, y) = \begin{cases} \frac{x^2 y}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Tasks

1. Compute $u_x(0, 0)$ from the definition.
2. Compute $u_y(0, 0)$ from the definition.
3. Compute formulas for u_x, u_y away from the origin and explain why they do not determine the values at the origin.

Question 9 – Solution

Strategy. At the exceptional point, use one-variable difference quotients along the coordinate axes.

Step 1: x -partial at the origin

$$u_x(0,0) = \lim_{h \rightarrow 0} \frac{u(h,0) - u(0,0)}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = \boxed{0}.$$

Step 2: y -partial at the origin

$$u_y(0,0) = \lim_{k \rightarrow 0} \frac{u(0,k) - u(0,0)}{k} = \lim_{k \rightarrow 0} \frac{0}{k} = \boxed{0}.$$

Step 3: Away from the origin Quotient differentiation gives

$$\boxed{u_x = \frac{2xy^3}{(x^2 + y^2)^2}}, \quad \boxed{u_y = \frac{x^2(x^2 - y^2)}{(x^2 + y^2)^2}} \quad ((x,y) \neq (0,0)).$$

These formulas were derived where the denominator is nonzero, so substituting $(0,0)$ would be invalid. The definition supplies the missing partial values.