

Question 10

Define

$$F(x, y) = \begin{cases} \frac{x^3 - 3xy^2}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ c, & (x, y) = (0, 0). \end{cases}$$

Tasks

1. Determine $\lim_{(x,y) \rightarrow (0,0)} F(x, y)$ using polar coordinates.
2. Find the unique value of c making F continuous at the origin.
3. Verify the estimate uniformly in angle.

Question 10 – Solution

Strategy. Recognize the cubic numerator's polar form and isolate one factor of radial distance.

Step 1: Convert With $x = r \cos \theta$ and $y = r \sin \theta$,

$$x^3 - 3xy^2 = r^3(\cos^3 \theta - 3 \cos \theta \sin^2 \theta) = r^3 \cos 3\theta.$$

Since $x^2 + y^2 = r^2$, for $r > 0$,

$$F(x, y) = r \cos 3\theta.$$

Step 2: Uniform estimate For every angle,

$$|F(x, y)| = r |\cos 3\theta| \leq r = \sqrt{x^2 + y^2} \longrightarrow 0.$$

Hence

$$\boxed{\lim_{(x,y) \rightarrow (0,0)} F(x, y) = 0}.$$

Step 3: Continuity value Continuity requires the assigned value to equal the limit, so the unique choice is

$$\boxed{c = 0}.$$

The bound is independent of θ , verifying that no directional or curved approach can change the result.