

Question 7

Evaluate

$$\lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2) \sin\left(\frac{1}{x^2 + y^2}\right).$$

Tasks

1. Explain why the oscillatory factor has no limit.
2. Bound the full expression.
3. Prove the limit despite the oscillation.

Question 7 – Solution

Strategy. The sine factor is bounded even though it oscillates; the radial factor forces the product to zero.

Step 1: Absolute bound Since $|\sin u| \leq 1$,

$$\left| (x^2 + y^2) \sin\left(\frac{1}{x^2 + y^2}\right) \right| \leq x^2 + y^2.$$

Step 2: Squeeze As $(x, y) \rightarrow (0, 0)$, the right side tends to 0. Therefore

$$\lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2) \sin\left(\frac{1}{x^2 + y^2}\right) = 0.$$

Step 3: Why oscillation is harmless Along radii for which $1/(x^2 + y^2)$ hits alternating sine peaks, the sine values do not settle. Nevertheless, the product's magnitude never exceeds the shrinking radius squared, so every path is controlled by the same bound.