

Question 5

For

$$f(x, y) = \frac{xy}{x^2 + y^2}, \quad (x, y) \neq (0, 0),$$

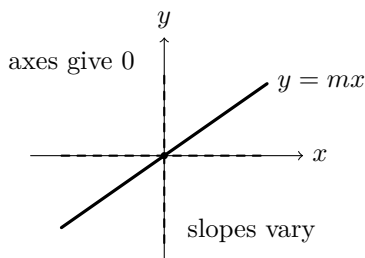
compare its two iterated limits with its joint limit at the origin.

Tasks

1. Compute both iterated limits.
2. Test the joint limit along $y = mx$.
3. Explain why the conclusions differ.

Question 5 – Solution

Strategy. Iterated limits approach along coordinate-aligned stages; the joint limit must agree along every path.



Step 1: Iterated limits For fixed $y \neq 0$, $\lim_{x \rightarrow 0} xy/(x^2 + y^2) = 0$, so

$$\lim_{y \rightarrow 0} \left(\lim_{x \rightarrow 0} f(x, y) \right) = 0.$$

Similarly, for fixed $x \neq 0$, the inner y -limit is 0, giving

$$\lim_{x \rightarrow 0} \left(\lim_{y \rightarrow 0} f(x, y) \right) = 0.$$

Step 2: Joint paths Along $y = mx$,

$$f(x, mx) = \frac{mx^2}{x^2 + m^2x^2} = \frac{m}{1 + m^2}.$$

This is 0 for $m = 0$ but $1/2$ for $m = 1$.

Conclusion. Both iterated limits equal $\boxed{0}$, yet the joint limit $\boxed{\text{does not exist}}$. Iterated limits inspect only two staged approaches and cannot control arbitrary simultaneous approaches.