

Question 4

Evaluate

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^2 + y^2}.$$

Tasks

1. Establish a useful bound without selecting paths.
2. Apply the squeeze theorem.
3. Give an independent polar-coordinate check.

Question 4 – Solution

Strategy. Bound the numerator using $x^2 \leq x^2 + y^2$ and then compare with a quantity tending to zero.

Step 1: Bound For $(x, y) \neq (0, 0)$,

$$0 \leq \frac{x^2 y^2}{x^2 + y^2} \leq \frac{(x^2 + y^2) y^2}{x^2 + y^2} = y^2.$$

As $(x, y) \rightarrow (0, 0)$, $y^2 \rightarrow 0$.

Step 2: Squeeze Both bounding functions tend to 0, hence

$$\boxed{\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^2 + y^2} = 0}.$$

Step 3: Polar check With $x = r \cos \theta$ and $y = r \sin \theta$,

$$\frac{x^2 y^2}{x^2 + y^2} = r^2 \cos^2 \theta \sin^2 \theta.$$

Because $0 \leq \cos^2 \theta \sin^2 \theta \leq 1/4$, its absolute value is at most $r^2/4 \rightarrow 0$, uniformly in direction.