

Question 10

Let $\mathbf{r} : [a, b] \rightarrow \mathbb{R}^3$ be continuously differentiable.

Tasks

1. Prove the chord–arc inequality

$$\|\mathbf{r}(b) - \mathbf{r}(a)\| \leq \int_a^b \|\mathbf{r}'(t)\| dt.$$

2. Give a clear geometric interpretation.
3. Show that equality holds if $\mathbf{r}'(t) = \lambda(t)\mathbf{u}$, where $\mathbf{u}$ is fixed unit vector and $\lambda(t) \geq 0$.

Question 10 – Solution

Strategy. Express displacement as the integral of the velocity vector and apply a dot-product estimate in the displacement direction.

Step 1: Zero-displacement case If $\mathbf{r}(b) = \mathbf{r}(a)$, the left side is zero and the inequality follows because the integral of speed is nonnegative.

Now suppose $\mathbf{d} = \mathbf{r}(b) - \mathbf{r}(a) \neq \mathbf{0}$ and define the unit vector $\mathbf{u} = \mathbf{d}/\|\mathbf{d}\|$.

Step 2: Use the Fundamental Theorem

$$\mathbf{d} = \int_a^b \mathbf{r}'(t) dt.$$

Dot both sides with $\mathbf{u}$:

$$\begin{aligned}\|\mathbf{d}\| &= \mathbf{d} \cdot \mathbf{u} \\ &= \int_a^b \mathbf{r}'(t) \cdot \mathbf{u} dt \\ &\leq \int_a^b |\mathbf{r}'(t) \cdot \mathbf{u}| dt \\ &\leq \int_a^b \|\mathbf{r}'(t)\| \|\mathbf{u}\| dt \\ &= \int_a^b \|\mathbf{r}'(t)\| dt.\end{aligned}$$

Therefore

$$\|\mathbf{r}(b) - \mathbf{r}(a)\| \leq \int_a^b \|\mathbf{r}'(t)\| dt.$$

Step 3: Geometry and equality The straight chord is the shortest connection between the endpoints; following a curved or reversing path cannot be shorter. If $\mathbf{r}' = \lambda \mathbf{u}$ with $\lambda \geq 0$, then

$$\mathbf{r}(b) - \mathbf{r}(a) = \mathbf{u} \int_a^b \lambda(t) dt,$$

so its norm is $\int_a^b \lambda(t) dt$. Meanwhile $\|\mathbf{r}'\| = \lambda$, giving equality. The condition describes travel in one fixed direction without backtracking.