

Question 8

Find the exact length of the semicubical parabola

$$\mathbf{r}(t) = \langle t^2, t^3, 0 \rangle, \quad 0 \leq t \leq 2.$$

Tasks

1. Factor the speed using the sign of t .
2. Evaluate the arc-length integral by substitution.
3. Explain why the zero velocity at $t = 0$ does not make the length integral improper.

Question 8 – Solution

Strategy. Factor t^2 from the squared speed and use $t \geq 0$ before integrating.

Step 1: Speed

$$\mathbf{r}'(t) = \langle 2t, 3t^2, 0 \rangle.$$

Hence

$$\begin{aligned}\|\mathbf{r}'(t)\| &= \sqrt{4t^2 + 9t^4} \\ &= \sqrt{t^2(4 + 9t^2)} \\ &= |t|\sqrt{4 + 9t^2}.\end{aligned}$$

On $[0, 2]$, $|t| = t$, so the speed is $t\sqrt{4 + 9t^2}$.

Step 2: Integrate Let $u = 4 + 9t^2$, so $du = 18t \, dt$:

$$\begin{aligned}L &= \int_0^2 t\sqrt{4 + 9t^2} \, dt \\ &= \frac{1}{18} \int_4^{40} u^{1/2} \, du \\ &= \frac{1}{18} \left[\frac{2}{3} u^{3/2} \right]_4^{40} \\ &= \frac{1}{27} (40^{3/2} - 4^{3/2}) \\ &= \boxed{\frac{80\sqrt{10} - 8}{27}}.\end{aligned}$$

Step 3: Endpoint regularity Although $\mathbf{r}'(0) = \mathbf{0}$, the speed $t\sqrt{4 + 9t^2}$ is continuous at 0 and equals zero there. A continuous integrand on a closed interval defines an ordinary proper integral; no singularity is present.