

Question 7

For

$$\mathbf{r}(t) = \left\langle t^2, \frac{1}{6}(4t+1)^{3/2}, 0 \right\rangle, \quad t \geq 0,$$

find the parameter value $T > 0$ at which the distance traveled from $t = 0$ first equals $35/4$.

Tasks

1. Derive the arc-length function $s(T)$.
2. Solve the resulting equation exactly.
3. Explain why the solution is unique.

Question 7 – Solution

Strategy. Differentiate carefully; the squared speed is a perfect square on the stated domain.

Step 1: Speed

$$\mathbf{r}'(t) = \langle 2t, \sqrt{4t+1}, 0 \rangle.$$

For $t \geq 0$,

$$\begin{aligned}\|\mathbf{r}'(t)\| &= \sqrt{4t^2 + 4t + 1} \\ &= \sqrt{(2t+1)^2} = |2t+1| = 2t+1.\end{aligned}$$

Step 2: Arc-length function

$$\begin{aligned}s(T) &= \int_0^T (2t+1) dt \\ &= [t^2 + t]_0^T = T^2 + T.\end{aligned}$$

Set this equal to the prescribed distance:

$$T^2 + T = \frac{35}{4}.$$

Multiplying by 4 and factoring,

$$4T^2 + 4T - 35 = (2T-5)(2T+7) = 0.$$

The candidates are $T = 5/2$ and $T = -7/2$; only the first lies in the domain. Hence

$$\boxed{T = \frac{5}{2}}.$$

Step 3: Uniqueness For $T > 0$, $s'(T) = 2T + 1 > 0$, so s is strictly increasing. Therefore the displayed equation has at most one positive solution. Since $s(0) = 0$ and $s(T) \rightarrow \infty$, it has exactly one. Direct substitution gives $s(5/2) = 25/4 + 10/4 = 35/4$, verifying the result.