

**Question 5**

The upper semicircle  $x^2 + y^2 = 4$  is represented by

$$\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, 0 \rangle, \quad 0 \leq t \leq \pi,$$

and also by

$$\mathbf{R}(u) = \langle 2 \cos(u^3), 2 \sin(u^3), 0 \rangle, \quad 0 \leq u \leq \sqrt[3]{\pi}.$$

**Tasks**

1. Compute the length using each parametrization.
2. Explain why the nonconstant speed of  $\mathbf{R}$  does not change the result.
3. Identify the substitution connecting the two integrals.

**Question 5 – Solution**

**Strategy.** Calculate both speed integrals, taking care that  $u \geq 0$  when simplifying an absolute value.

**Step 1: Standard parametrization**

$$\mathbf{r}'(t) = \langle -2 \sin t, 2 \cos t, 0 \rangle, \quad \|\mathbf{r}'(t)\| = 2.$$

Hence

$$L_r = \int_0^\pi 2 \, dt = \boxed{2\pi}.$$

**Step 2: Cubic reparametrization**

$$\mathbf{R}'(u) = \langle -6u^2 \sin(u^3), 6u^2 \cos(u^3), 0 \rangle.$$

Because  $u \geq 0$ ,

$$\|\mathbf{R}'(u)\| = 6u^2.$$

Therefore

$$L_R = \int_0^{\sqrt[3]{\pi}} 6u^2 \, du = [2u^3]_0^{\sqrt[3]{\pi}} = \boxed{2\pi}.$$

**Step 3: Invariance** The substitution  $t = u^3$  gives  $dt = 3u^2 du$ , so

$$2 \, dt = 6u^2 \, du.$$

The second parametrization changes the rate at which points are visited, but it is increasing and covers the same semicircle once. The speed factor exactly compensates for the parameter distortion.