

Question 3

Determine the exact length of

$$\mathbf{r}(t) = \left\langle t - \frac{t^3}{3}, t^2, t + \frac{t^3}{3} \right\rangle, \quad 0 \leq t \leq 2.$$

Tasks

1. Compute the speed without prematurely expanding every square.
2. Identify and justify the perfect-square simplification.
3. Evaluate the length and verify the speed is never negative.

Question 3 – Solution

Strategy. Differentiate first and group symmetric components; the radicals collapse to a polynomial.

Step 1: Differentiate

$$\mathbf{r}'(t) = \langle 1 - t^2, 2t, 1 + t^2 \rangle.$$

Thus

$$\begin{aligned}\|\mathbf{r}'(t)\|^2 &= (1 - t^2)^2 + (2t)^2 + (1 + t^2)^2 \\ &= (1 - 2t^2 + t^4) + 4t^2 + (1 + 2t^2 + t^4) \\ &= 2 + 4t^2 + 2t^4 \\ &= 2(1 + t^2)^2.\end{aligned}$$

Step 2: Take the square root

$$\|\mathbf{r}'(t)\| = \sqrt{2}|1 + t^2|.$$

Since $1 + t^2 > 0$ for every real t , this becomes

$$\|\mathbf{r}'(t)\| = \sqrt{2}(1 + t^2).$$

Step 3: Integrate

$$\begin{aligned}L &= \sqrt{2} \int_0^2 (1 + t^2) dt \\ &= \sqrt{2} \left[t + \frac{t^3}{3} \right]_0^2 \\ &= \boxed{\frac{14\sqrt{2}}{3}}.\end{aligned}$$

The speed is strictly positive because $1 + t^2 > 0$, so the curve is regular throughout the interval.