

Question 1

Find the exact length of

$$\mathbf{r}(t) = \left\langle 3t, 4t, \frac{1}{2}t^2 \right\rangle, \quad 0 \leq t \leq 2.$$

Tasks

1. Compute the speed and simplify it completely.
2. Set up and evaluate the arc-length integral.
3. Check that the result exceeds the straight-line distance between the endpoints.

Question 1 – Solution

Strategy. Arc length is the integral of speed: $L = \int_a^b \|\mathbf{r}'(t)\| dt$.

Step 1: Velocity and speed

$$\mathbf{r}'(t) = \langle 3, 4, t \rangle.$$

Therefore

$$\|\mathbf{r}'(t)\| = \sqrt{3^2 + 4^2 + t^2} = \sqrt{25 + t^2}.$$

Step 2: Integrate Use

$$\int \sqrt{t^2 + a^2} dt = \frac{1}{2} \left(t\sqrt{t^2 + a^2} + a^2 \ln \left| t + \sqrt{t^2 + a^2} \right| \right).$$

With $a = 5$,

$$\begin{aligned} L &= \int_0^2 \sqrt{25 + t^2} dt \\ &= \frac{1}{2} \left[t\sqrt{25 + t^2} + 25 \ln \left(t + \sqrt{25 + t^2} \right) \right]_0^2 \\ &= \boxed{\sqrt{29} + \frac{25}{2} \ln \left(\frac{2 + \sqrt{29}}{5} \right)}. \end{aligned}$$

Step 3: Endpoint check The displacement is

$$\mathbf{r}(2) - \mathbf{r}(0) = \langle 6, 8, 2 \rangle,$$

whose length is $\sqrt{36 + 64 + 4} = 2\sqrt{26} \approx 10.198$. The arc length is approximately 10.261, which is slightly larger, as a nonstraight path requires.