

Question 10

At a point $P = (1, -1, 2)$ on a regular curve, a unit tangent and a candidate principal normal are

$$\mathbf{T} = \frac{1}{3} \langle 2, 1, 2 \rangle, \quad \mathbf{N} = \frac{1}{3\sqrt{5}} \langle -4, 7, \frac{1}{2} \rangle.$$

Tasks

1. Determine whether the supplied vectors can belong to a Frenet frame, and diagnose any defect.
2. Replace $\mathbf{N}$ by the unit vector in the direction of the component of $\langle -4, 7, 1 \rangle$ perpendicular to $\mathbf{T}$.
3. Find the corrected $\mathbf{B}$ and the osculating plane at P .

Question 10 – Solution

Strategy. First audit unit length and orthogonality. Then use vector projection to remove the tangential component from the proposed bending direction.

Step 1: Diagnose the supplied normal The tangent is unit because $(4 + 1 + 4)/9 = 1$. But

$$\mathbf{T} \cdot \mathbf{N} = \frac{2(-4) + 1(7) + 2(1/2)}{9\sqrt{5}} = 0,$$

while

$$\|\mathbf{N}\|^2 = \frac{16 + 49 + 1/4}{45} = \frac{29}{20} \neq 1.$$

Thus it is perpendicular to $\mathbf{T}$ but is not a unit vector, so the data are not a Frenet frame.

Step 2: Correct the bending direction Let $\mathbf{w} = \langle -4, 7, 1 \rangle$. Its tangential scalar component is

$$\mathbf{w} \cdot \mathbf{T} = \frac{-8 + 7 + 2}{3} = \frac{1}{3}.$$

Therefore

$$\mathbf{w}_{\perp} = \mathbf{w} - (\mathbf{w} \cdot \mathbf{T})\mathbf{T} = \langle -4, 7, 1 \rangle - \frac{1}{9} \langle 2, 1, 2 \rangle = \frac{1}{9} \langle -38, 62, 7 \rangle.$$

Since $\sqrt{(-38)^2 + 62^2 + 7^2} = \sqrt{5337} = 3\sqrt{593}$,

$$\hat{\mathbf{N}} = \frac{1}{3\sqrt{593}} \langle -38, 62, 7 \rangle.$$

Step 3: Binormal and osculating plane

$$\mathbf{B} = \mathbf{T} \times \hat{\mathbf{N}} = \frac{1}{\sqrt{593}} \langle -13, -10, 18 \rangle.$$

The numerator has length $\sqrt{169 + 100 + 324} = \sqrt{593}$. Since $\mathbf{B}$ is normal to the osculating plane through P ,

$$-13(x - 1) - 10(y + 1) + 18(z - 2) = 0,$$

or

$$13x + 10y - 18z + 33 = 0.$$

Substitution of P gives zero, and the construction guarantees a right-handed orthonormal frame.