

Question 9

Let

$$\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, t \rangle.$$

Find all $t \in [0, 2\pi)$ for which the unit tangent is perpendicular to the fixed vector $\mathbf{a} = \langle 1, 1, 0 \rangle$, and find the corresponding points.

Tasks

1. Explain why it is enough to test $\mathbf{r}'(t) \cdot \mathbf{a} = 0$.
2. Solve the resulting trigonometric equation exactly.
3. Evaluate the curve at every admissible parameter.

Question 9 – Solution

Strategy. Normalizing a nonzero vector multiplies it by a positive scalar, which does not change whether its dot product with $\mathbf{a}$ is zero.

Step 1: Tangent direction

$$\mathbf{r}'(t) = \langle -2 \sin t, 2 \cos t, 1 \rangle, \quad \|\mathbf{r}'(t)\| = \sqrt{5}.$$

Thus $\mathbf{T} \cdot \mathbf{a} = 0$ exactly when $\mathbf{r}' \cdot \mathbf{a} = 0$.

Step 2: Solve

$$\begin{aligned} \mathbf{r}'(t) \cdot \mathbf{a} &= -2 \sin t + 2 \cos t = 0 \\ \implies \sin t &= \cos t \\ \implies \tan t &= 1. \end{aligned}$$

On $[0, 2\pi)$,

$$t = \frac{\pi}{4}, \frac{5\pi}{4}.$$

Neither solution is lost by using tangent because $\cos t \neq 0$ at both values.

Step 3: Points

$$\mathbf{r}\left(\frac{\pi}{4}\right) = \left\langle \sqrt{2}, \sqrt{2}, \frac{\pi}{4} \right\rangle,$$

and

$$\mathbf{r}\left(\frac{5\pi}{4}\right) = \left\langle -\sqrt{2}, -\sqrt{2}, \frac{5\pi}{4} \right\rangle.$$

At either parameter, substituting into $-2 \sin t + 2 \cos t$ gives zero, verifying perpendicularity.