

Question 8

Consider

$$\mathbf{r}(t) = \langle t, t^3, 0 \rangle.$$

Tasks

1. Find $\mathbf{T}(0)$.
2. Determine whether the principal normal and binormal are defined at $t = 0$ by the standard Frenet formulas.
3. Compute the one-sided limiting normal directions and explain the geometric issue.

Question 8 – Solution

Strategy. Regularity of $\mathbf{r}$ guarantees $\mathbf{T}$, but defining $\mathbf{N}$ additionally requires $\mathbf{T}' \neq \mathbf{0}$.

Step 1: Unit tangent

$$\mathbf{r}'(t) = \langle 1, 3t^2, 0 \rangle, \quad \mathbf{T}(t) = \frac{\langle 1, 3t^2, 0 \rangle}{\sqrt{1 + 9t^4}}.$$

Thus $\boxed{\mathbf{T}(0) = \langle 1, 0, 0 \rangle}$.

Step 2: Differentiate $\mathbf{T}$ Differentiation and simplification give

$$\mathbf{T}'(t) = \frac{1}{(1 + 9t^4)^{3/2}} \langle -18t^3, 6t, 0 \rangle = \frac{6t}{(1 + 9t^4)^{3/2}} \langle -3t^2, 1, 0 \rangle.$$

Therefore $\mathbf{T}'(0) = \mathbf{0}$. The expression

$$\mathbf{N}(0) = \frac{\mathbf{T}'(0)}{\|\mathbf{T}'(0)\|}$$

would divide by zero, so $\boxed{\mathbf{N}(0) \text{ and } \mathbf{B}(0) \text{ are not defined by the Frenet formulas}}$.

Step 3: One-sided behavior For $t > 0$, normalization yields a vector tending to $\langle 0, 1, 0 \rangle$. For $t < 0$, the scalar factor $6t$ reverses direction, giving the limit $\langle 0, -1, 0 \rangle$. The unequal limits show that the preferred bending direction flips at the inflection point. Although the curve has a tangent there, it has no single principal-normal direction.