

Question 7

For

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle,$$

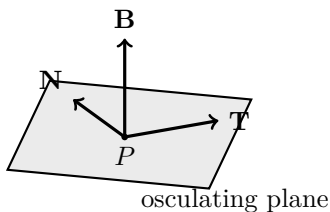
find equations of the osculating plane and normal plane at $t = 1$.

Tasks

1. Compute suitable directions for $\mathbf{T}$ and $\mathbf{B}$.
2. Use $\mathbf{B}$ as the osculating-plane normal.
3. Use $\mathbf{T}$ as the normal-plane normal and verify both planes pass through the curve point.

Question 7 – Solution

Strategy. The osculating plane is spanned by $\mathbf{T}$ and $\mathbf{N}$, so its normal is $\mathbf{B}$. The normal plane is perpendicular to $\mathbf{T}$.

**Step 1: Point and tangent direction**

$$P = \mathbf{r}(1) = (1, 1, 1), \quad \mathbf{r}'(1) = \langle 1, 2, 3 \rangle.$$

We may use $\langle 1, 2, 3 \rangle$ as the normal vector of the normal plane.

Step 2: Binormal direction

$$\mathbf{r}''(t) = \langle 0, 2, 6t \rangle, \quad \mathbf{r}''(1) = \langle 0, 2, 6 \rangle.$$

Then

$$\mathbf{r}'(1) \times \mathbf{r}''(1) = \langle 1, 2, 3 \rangle \times \langle 0, 2, 6 \rangle = \langle 6, -6, 2 \rangle = 2 \langle 3, -3, 1 \rangle.$$

Hence an osculating-plane normal is $\langle 3, -3, 1 \rangle$.

Step 3: Plane equations Through $P = (1, 1, 1)$, the osculating plane is

$$3(x - 1) - 3(y - 1) + (z - 1) = 0 \implies \boxed{3x - 3y + z = 1}.$$

The normal plane is

$$(x - 1) + 2(y - 1) + 3(z - 1) = 0 \implies \boxed{x + 2y + 3z = 6}.$$

Substitution of $(1, 1, 1)$ verifies both equations, and their normals have the required Frenet directions.