

Question 6

At a regular point of a curve, suppose

$$\mathbf{r}'(t_0) = \langle 1, 2, 2 \rangle, \quad \mathbf{r}''(t_0) = \langle 2, 0, 1 \rangle.$$

Tasks

1. Find the unit tangent $\mathbf{T}(t_0)$.
2. Find the binormal using $\mathbf{r}' \times \mathbf{r}''$.
3. Obtain the principal normal from $\mathbf{N} = \mathbf{B} \times \mathbf{T}$ and verify the frame.

Question 6 – Solution

Strategy. The cross product $\mathbf{r}' \times \mathbf{r}''$ gives the binormal direction without differentiating a normalized vector.

Step 1: Tangent

$$\|\mathbf{r}'(t_0)\| = \sqrt{1+4+4} = 3, \quad \boxed{\mathbf{T}(t_0) = \frac{1}{3} \langle 1, 2, 2 \rangle}.$$

Step 2: Binormal direction

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 2 & 0 & 1 \end{vmatrix} = \langle 2, 3, -4 \rangle.$$

Its length is $\sqrt{4+9+16} = \sqrt{29}$, so

$$\boxed{\mathbf{B}(t_0) = \frac{1}{\sqrt{29}} \langle 2, 3, -4 \rangle}.$$

Step 3: Principal normal In a right-handed frame, $\mathbf{B} \times \mathbf{T} = \mathbf{N}$. Thus

$$\begin{aligned} \mathbf{N}(t_0) &= \frac{1}{3\sqrt{29}} \langle 2, 3, -4 \rangle \times \langle 1, 2, 2 \rangle \\ &= \boxed{\frac{1}{3\sqrt{29}} \langle 14, -8, 1 \rangle}. \end{aligned}$$

Its squared numerator length is $196 + 64 + 1 = 261 = 9(29)$, so it has unit length. Direct dot products with $\mathbf{T}$ and $\mathbf{B}$ are zero, verifying the frame.