

Question 5

The curve

$$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$$

is reparametrized by $t = 2u + 1$. Let $\mathbf{R}(u) = \mathbf{r}(2u + 1)$.

Tasks

1. Compare the unit tangents of $\mathbf{r}$ at $t = 2u + 1$ and $\mathbf{R}$ at u .
2. Compare their principal normals.
3. State what changes if the reparametrization is instead $t = 1 - 2u$.

Question 5 – Solution

Strategy. Apply the chain rule and keep track of the sign of the parameter derivative during normalization.

Step 1: Increasing reparametrization The chain rule gives

$$\mathbf{R}'(u) = 2\mathbf{r}'(2u + 1).$$

Since the factor 2 is positive,

$$\frac{\mathbf{R}'(u)}{\|\mathbf{R}'(u)\|} = \frac{2\mathbf{r}'(2u + 1)}{2\|\mathbf{r}'(2u + 1)\|} = \boxed{\mathbf{T}_r(2u + 1)}.$$

Thus an increasing reparametrization preserves the unit tangent.

Step 2: Principal normal Differentiating $\mathbf{T}_R(u) = \mathbf{T}_r(2u + 1)$ gives

$$\mathbf{T}'_R(u) = 2\mathbf{T}'_r(2u + 1).$$

Again the positive factor cancels upon normalization, so

$$\boxed{\mathbf{N}_R(u) = \mathbf{N}_r(2u + 1)}.$$

Step 3: Reversed parameter If $t = 1 - 2u$, then $dt/du = -2$. Consequently

$$\boxed{\mathbf{T}_R = -\mathbf{T}_r},$$

at corresponding points. Differentiating this relation with respect to u introduces a second negative factor, so $\mathbf{T}'_R = 2\mathbf{T}'_r$ and

$$\boxed{\mathbf{N}_R = \mathbf{N}_r}.$$

Therefore $\mathbf{B}_R = \mathbf{T}_R \times \mathbf{N}_R = -\mathbf{B}_r$. Reversing orientation changes $\mathbf{T}$ and $\mathbf{B}$, but not the principal normal.