

Question 4

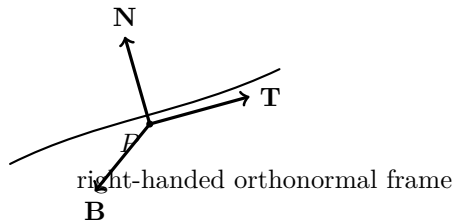
Let $\mathbf{r}$ be a regular curve for which $\mathbf{T}'(t) \neq \mathbf{0}$.

Tasks

1. Prove that $\mathbf{T}(t) \cdot \mathbf{N}(t) = 0$.
2. Prove that $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ is a unit vector.
3. Explain why $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ is a right-handed orthonormal frame.

Question 4 – Solution

Strategy. Differentiate the invariant $\mathbf{T} \cdot \mathbf{T} = 1$, then use standard cross-product identities.



Step 1: Tangent-normal orthogonality Since $\mathbf{T}$ is a unit vector,

$$\mathbf{T}(t) \cdot \mathbf{T}(t) = 1.$$

Differentiate using the dot-product rule:

$$\mathbf{T}' \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{T}' = 0 \implies 2\mathbf{T} \cdot \mathbf{T}' = 0.$$

Because $\mathbf{N} = \mathbf{T}' / \|\mathbf{T}'\|$ and the denominator is nonzero,

$$\boxed{\mathbf{T} \cdot \mathbf{N} = 0}.$$

Step 2: Length of $\mathbf{B}$ For vectors separated by angle θ ,

$$\|\mathbf{T} \times \mathbf{N}\| = \|\mathbf{T}\| \|\mathbf{N}\| \sin \theta.$$

Here both lengths are 1 and $\theta = \pi/2$, so

$$\boxed{\|\mathbf{B}\| = 1}.$$

The cross product is perpendicular to both factors; hence $\mathbf{B}$ is orthogonal to both $\mathbf{T}$ and $\mathbf{N}$.

Step 3: Orientation The ordered relation $\mathbf{T} \times \mathbf{N} = \mathbf{B}$ is exactly the right-hand rule. Thus the three mutually perpendicular unit vectors form a right-handed orthonormal frame.