

Question 10

Find every differentiable vector function $\mathbf{r}(t)$ satisfying

$$\mathbf{r}'(t) = 2t\mathbf{r}(t), \quad \mathbf{r}(0) = \langle a, b, c \rangle,$$

where a, b, c are constants. Then prove directly that any two solutions with the same initial vector must be identical.

Tasks

1. Solve the vector initial-value problem component by component.
2. Verify the resulting formula by differentiation and substitution.
3. Prove uniqueness using an integrating factor applied to the difference of two solutions.

Question 10 – Solution

Strategy. The vector equation is three identical scalar equations. Solve them componentwise, then prove uniqueness with an integrating factor.

Step 1: Split into components Write $\mathbf{r} = \langle x, y, z \rangle$. Then

$$x' = 2tx, \quad y' = 2ty, \quad z' = 2tz.$$

For a nonzero component w , separation gives

$$\frac{w'}{w} = 2t \implies \ln |w| = t^2 + C \implies w = Ke^{t^2}.$$

The formula also includes the zero solution when $K = 0$.

Step 2: Apply the initial vector Since $e^0 = 1$, the constants are a, b, c . Thus

$$\boxed{\mathbf{r}(t) = e^{t^2} \langle a, b, c \rangle}.$$

Indeed,

$$\mathbf{r}'(t) = 2te^{t^2} \langle a, b, c \rangle = 2t\mathbf{r}(t),$$

and $\mathbf{r}(0) = \langle a, b, c \rangle$.

Step 3: Direct uniqueness proof Suppose $\mathbf{r}_1$ and $\mathbf{r}_2$ share the initial value, and let $\mathbf{d} = \mathbf{r}_1 - \mathbf{r}_2$. Then

$$\mathbf{d}' = 2t\mathbf{d}, \quad \mathbf{d}(0) = \mathbf{0}.$$

Multiply by e^{-t^2} and use the product rule:

$$\frac{d}{dt}(e^{-t^2} \mathbf{d}(t)) = -2te^{-t^2} \mathbf{d} + e^{-t^2} \mathbf{d}' = \mathbf{0}.$$

Hence $e^{-t^2} \mathbf{d}$ is constant and equals its value $\mathbf{0}$ at $t = 0$. Therefore $\mathbf{d} = \mathbf{0}$, proving $\boxed{\mathbf{r}_1 = \mathbf{r}_2}$.