

**Question 9**

Let

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle, \quad t \geq 0,$$

and reparametrize it by  $t = s^2 + 1$ .

**Tasks**

1. Write the composite vector function  $\mathbf{R}(s) = \mathbf{r}(s^2 + 1)$ .
2. Compute  $\mathbf{R}'(s)$  directly.
3. Compute it again using the vector chain rule and reconcile the fact that  $\mathbf{R}'(0) = \mathbf{0}$  while  $\mathbf{r}'(1) \neq \mathbf{0}$ .

**Question 9 – Solution**

**Strategy.** Compare direct componentwise differentiation with  $(\mathbf{r} \circ g)' = \mathbf{r}'(g)g'$ .

**Step 1: Composition**

$$\mathbf{R}(s) = \langle s^2 + 1, (s^2 + 1)^2, (s^2 + 1)^3 \rangle.$$

**Step 2: Direct differentiation**

$$\begin{aligned}\mathbf{R}'(s) &= \langle 2s, 2(s^2 + 1)(2s), 3(s^2 + 1)^2(2s) \rangle \\ &= \langle 2s, 4s(s^2 + 1), 6s(s^2 + 1)^2 \rangle.\end{aligned}$$

**Step 3: Chain-rule check** Since

$$\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle, \quad g(s) = s^2 + 1,$$

we obtain

$$\begin{aligned}\mathbf{R}'(s) &= \mathbf{r}'(g(s))g'(s) \\ &= \langle 1, 2(s^2 + 1), 3(s^2 + 1)^2 \rangle (2s),\end{aligned}$$

which is identical to the direct result.

At  $s = 0$ ,  $g'(0) = 0$ , so  $\mathbf{R}'(0) = \mathbf{0}$  even though

$$\mathbf{r}'(1) = \langle 1, 2, 3 \rangle \neq \mathbf{0}.$$

There is no contradiction: the new parameter momentarily stops changing the old parameter. The geometric curve is regular at  $t = 1$ , but this particular parametrization is not regular at  $s = 0$ .