

Question 8

Let

$$\mathbf{F}(t) = \int_1^{t^2} \left\langle e^u, \frac{1}{u}, \cos(\pi u) \right\rangle du, \quad t > 0.$$

Tasks

1. Find $\mathbf{F}'(t)$ without first evaluating the integral.
2. Evaluate $\mathbf{F}'(1)$ and $\mathbf{F}(1)$.
3. Explain the roles of the Fundamental Theorem of Calculus and the chain rule.

Question 8 – Solution

Strategy. Differentiate the vector integral componentwise. The moving upper limit contributes its derivative by the chain rule.

Step 1: General rule If

$$\mathbf{F}(t) = \int_a^{g(t)} \mathbf{f}(u) du,$$

then the Fundamental Theorem gives the derivative with respect to the upper limit, while the chain rule multiplies by $g'(t)$:

$$\mathbf{F}'(t) = \mathbf{f}(g(t))g'(t).$$

Step 2: Apply the rule Here $g(t) = t^2$, so $g'(t) = 2t$. Therefore

$$\begin{aligned}\mathbf{F}'(t) &= 2t \left\langle e^{t^2}, \frac{1}{t^2}, \cos(\pi t^2) \right\rangle \\ &= \left\langle 2te^{t^2}, \frac{2}{t}, 2t \cos(\pi t^2) \right\rangle.\end{aligned}$$

The restriction $t > 0$ ensures the interval between 1 and t^2 never crosses the singularity $u = 0$.

Step 3: Values at $t = 1$

$$\mathbf{F}'(1) = \langle 2e, 2, -2 \rangle.$$

Also, an integral with identical limits is the zero vector:

$$\mathbf{F}(1) = \int_1^1 \mathbf{f}(u) du = \langle 0, 0, 0 \rangle.$$

These results distinguish accumulated value from its instantaneous rate of change.