

Question 7

Define

$$\mathbf{r}(t) = \begin{cases} \langle t, t^2, t^3 \rangle, & t \leq 0, \\ \langle t, at, t^2 \rangle, & t > 0, \end{cases}$$

where a is a real constant.

Tasks

1. Determine all a for which $\mathbf{r}$ is continuous at 0.
2. Determine all a for which $\mathbf{r}$ is differentiable at 0.
3. For the differentiable case, find $\mathbf{r}'(0)$.

Question 7 – Solution

Strategy. Continuity compares one-sided function values; differentiability compares one-sided vector difference quotients.

Step 1: Continuity The defined value from the first branch is

$$\mathbf{r}(0) = \langle 0, 0, 0 \rangle.$$

From the left, $\langle t, t^2, t^3 \rangle \rightarrow \langle 0, 0, 0 \rangle$. From the right,

$$\langle t, at, t^2 \rangle \rightarrow \langle 0, 0, 0 \rangle$$

for every fixed real a . Therefore $\mathbf{r}$ is continuous at 0 for every real value of a .

Step 2: Left derivative As h approaches zero through negative values,

$$\frac{\mathbf{r}(h) - \mathbf{r}(0)}{h} = \langle 1, h, h^2 \rangle \longrightarrow \langle 1, 0, 0 \rangle.$$

Step 3: Right derivative As h approaches zero through positive values,

$$\frac{\mathbf{r}(h) - \mathbf{r}(0)}{h} = \langle 1, a, h \rangle \longrightarrow \langle 1, a, 0 \rangle.$$

The one-sided derivatives agree exactly when $a = 0$. Hence

$$\boxed{a = 0}, \quad \boxed{\mathbf{r}'(0) = \langle 1, 0, 0 \rangle}.$$

This also shows explicitly why continuity alone does not guarantee differentiability.