

Question 6

Suppose a differentiable vector function $\mathbf{r}(t)$ satisfies

$$\|\mathbf{r}(t)\| = 7$$

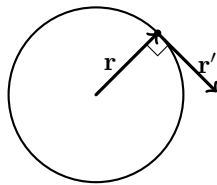
for every t in an interval.

Tasks

1. Prove that $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$ throughout the interval.
2. Explain the geometric meaning when $\mathbf{r}'(t) \neq \mathbf{0}$.
3. Show that $\mathbf{r} \cdot \mathbf{r}' = 0$ on an interval implies constant length; explain what extra datum fixes that length at 7.

Question 6 – Solution

Strategy. Square the norm to avoid differentiating a square root, then use the dot-product rule.



constant distance $\Rightarrow$ perpendicular change

Step 1: Differentiate the invariant The hypothesis gives

$$\mathbf{r}(t) \cdot \mathbf{r}(t) = \|\mathbf{r}(t)\|^2 = 49.$$

Differentiating both sides,

$$\mathbf{r}' \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{r}' = 0.$$

Because the dot product is commutative, the two terms are equal. Hence

$$2\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \quad \Longrightarrow \quad \boxed{\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0}.$$

Step 2: Geometry The position vector points radially from the origin. When $\mathbf{r}'(t) \neq \mathbf{0}$, it points along the tangent to the curve. Their zero dot product means the tangent is perpendicular to the radius, as expected for motion constrained to a sphere.

Step 3: Converse Now assume $\mathbf{r} \cdot \mathbf{r}' = 0$. Then

$$\frac{d}{dt} \|\mathbf{r}(t)\|^2 = \frac{d}{dt} (\mathbf{r} \cdot \mathbf{r}) = 2\mathbf{r} \cdot \mathbf{r}' = 0.$$

Therefore $\|\mathbf{r}(t)\|^2$ is constant on the interval, so its nonnegative square root $\|\mathbf{r}(t)\|$ is also constant. The constant is 7 if $\|\mathbf{r}(t_0)\| = 7$ at one point of the interval.