

Question 5

Evaluate exactly

$$\int_0^{\pi/2} \langle t \cos t, t \sin t, \sin t \cos t \rangle dt.$$

Tasks

1. Evaluate all three components exactly, showing each scalar integration step.
2. State the resulting vector.
3. Verify the third component using a double-angle identity.

Question 5 – Solution

Strategy. Integrate componentwise. The first two components require integration by parts; the third can be checked in two different ways.

Step 1: First component Integration by parts with $u = t$, $dv = \cos t \, dt$ gives

$$\begin{aligned}\int_0^{\pi/2} t \cos t \, dt &= [t \sin t]_0^{\pi/2} - \int_0^{\pi/2} \sin t \, dt \\ &= \frac{\pi}{2} - 1.\end{aligned}$$

Step 2: Second component With $u = t$, $dv = \sin t \, dt$,

$$\int_0^{\pi/2} t \sin t \, dt = [-t \cos t]_0^{\pi/2} + \int_0^{\pi/2} \cos t \, dt = 1.$$

Step 3: Third component Since $d(\sin^2 t)/dt = 2 \sin t \cos t$,

$$\int_0^{\pi/2} \sin t \cos t \, dt = \left[\frac{1}{2} \sin^2 t \right]_0^{\pi/2} = \frac{1}{2}.$$

Therefore

$$\boxed{\int_0^{\pi/2} \langle t \cos t, t \sin t, \sin t \cos t \rangle \, dt = \left\langle \frac{\pi}{2} - 1, 1, \frac{1}{2} \right\rangle}.$$

Using $\sin t \cos t = \frac{1}{2} \sin 2t$ gives the same third component: $[-\frac{1}{4} \cos 2t]_0^{\pi/2} = 1/2$.