

Question 1

For $t \neq 0$, let

$$\mathbf{r}(t) = \left\langle \frac{\sin 3t}{t}, \frac{e^{2t} - 1}{t}, \frac{1 - \cos t}{t^2} \right\rangle.$$

Tasks

1. Compute $\lim_{t \rightarrow 0} \mathbf{r}(t)$.
2. Define $\mathbf{r}(0)$ so that $\mathbf{r}$ is continuous at 0.
3. Justify why a vector limit exists precisely when all component limits exist.

Question 1 – Solution

Strategy. A vector limit is evaluated component by component. Rewrite each component in terms of a standard scalar limit.

Step 1: First component

$$\frac{\sin 3t}{t} = 3 \frac{\sin 3t}{3t} \longrightarrow 3(1) = 3.$$

Step 2: Second component Put $u = 2t$. Then

$$\frac{e^{2t} - 1}{t} = 2 \frac{e^u - 1}{u} \longrightarrow 2.$$

Step 3: Third component Multiply by the conjugate:

$$\begin{aligned} \frac{1 - \cos t}{t^2} &= \frac{(1 - \cos t)(1 + \cos t)}{t^2(1 + \cos t)} \\ &= \frac{\sin^2 t}{t^2(1 + \cos t)} = \left(\frac{\sin t}{t} \right)^2 \frac{1}{1 + \cos t} \longrightarrow \frac{1}{2}. \end{aligned}$$

Therefore

$$\boxed{\lim_{t \rightarrow 0} \mathbf{r}(t) = \langle 3, 2, \tfrac{1}{2} \rangle}.$$

Step 4: Continuous extension Define $\boxed{\mathbf{r}(0) = \langle 3, 2, \tfrac{1}{2} \rangle}$. This makes the function value equal its limit. The vector limit criterion follows because the distance to a candidate vector tends to zero exactly when each coordinate difference tends to zero.