

Question 10

Consider the closed space curve

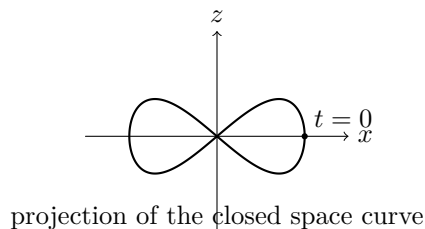
$$\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, \sin t \cos t \rangle, \quad 0 \leq t < 2\pi.$$

Tasks

1. Find two surfaces whose intersection contains the entire trace.
2. Determine all points where the trace meets the plane $z = 0$.
3. Prove that the parametrization traces each point exactly once, despite the repeated values of the z -component.

Question 10 – Solution

Strategy. Eliminate the parameter using the first two components. For uniqueness, retain the ordered pair $(\cos t, \sin t)$ rather than examining z alone.



Step 1: Eliminate the parameter The first two components give

$$x^2 + y^2 = 4(\cos^2 t + \sin^2 t) = 4.$$

Also,

$$xy = (2 \cos t)(2 \sin t) = 4 \sin t \cos t = 4z,$$

so $z = xy/4$. The trace is therefore contained in

$$\boxed{x^2 + y^2 = 4, \quad z = \frac{xy}{4}}.$$

Conversely, every point of this intersection has $(x/2, y/2)$ on the unit circle and hence equals $(\cos t, \sin t)$ for some t ; thus these equations describe the full trace.

Step 2: Intersections with $z = 0$ Since $z = xy/4$, we need $x = 0$ or $y = 0$. On $x^2 + y^2 = 4$, this yields

$$\boxed{(2, 0, 0), (-2, 0, 0), (0, 2, 0), (0, -2, 0)}.$$

Step 3: One-to-one traversal If $\mathbf{r}(t_1) = \mathbf{r}(t_2)$, then the first two coordinates imply

$$(\cos t_1, \sin t_1) = (\cos t_2, \sin t_2).$$

Thus $t_1 - t_2 = 2\pi k$ for an integer k . Inside $[0, 2\pi)$, only $k = 0$ is possible, so $t_1 = t_2$. Hence no point is repeated.