

Question 6

The vector functions

$$\mathbf{r}(t) = \langle \cos t, \sin t, \cos 2t \rangle, \quad 0 \leq t < 2\pi,$$

and

$$\mathbf{q}(u) = \langle \cos u, -\sin u, \cos 2u \rangle, \quad 0 \leq u < 2\pi,$$

are given.

1. Prove that they have the same geometric trace.
2. Compare their orientations starting at $(1, 0, 1)$.
3. Decide whether either parametrization visits a point twice on its stated interval.

Question 6 – Solution

Strategy. Find a parameter substitution connecting the two formulas, then use the first two components to test repeated points.

Step 1: Same trace For $0 < u < 2\pi$, put $t = 2\pi - u$; at $u = 0$, use $t = 0$. For the interior parameters,

$$\cos(2\pi - u) = \cos u, \quad \sin(2\pi - u) = -\sin u,$$

and

$$\cos(2t) = \cos(4\pi - 2u) = \cos 2u.$$

Hence $\mathbf{r}(2\pi - u) = \mathbf{q}(u)$, so every point of one trace occurs on the other. Thus their geometric traces are identical.

Step 2: Orientation Both begin at $(1, 0, 1)$. For small positive t , the second component of $\mathbf{r}(t)$ is positive. For small positive u , the second component of $\mathbf{q}(u)$ is negative. Therefore they traverse the trace in opposite orientations.

Step 3: Repeated-point test Suppose $\mathbf{r}(t_1) = \mathbf{r}(t_2)$. Its first two coordinates give

$$(\cos t_1, \sin t_1) = (\cos t_2, \sin t_2),$$

so $t_1 - t_2$ is an integer multiple of 2π . Since both parameters lie in $[0, 2\pi)$, this forces $t_1 = t_2$. The same argument applies to $\mathbf{q}$.

Thus

each parametrization traces the curve exactly once, in opposite directions.
