

Question 10

An open-top box has length x , width y , and height h . A design constraint fixes the sum of these dimensions:

$$x + y + h = 12, \quad x, y, h > 0.$$

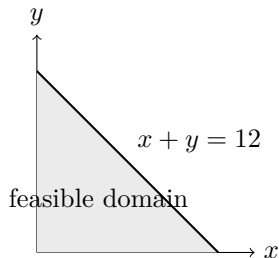
Its volume is $V(x, y) = xy(12 - x - y)$.

Tasks

1. State the natural domain.
2. Find the greatest possible volume without calculus.
3. Prove the maximizing dimensions are unique.

Question 10 – Solution

Strategy Interpret the three positive factors x , y , and $h = 12 - x - y$, then apply AM–GM.



Step 1: Domain Positivity requires

$$\boxed{x > 0, \quad y > 0, \quad x + y < 12}.$$

This is the interior of a triangular region.

Step 2: Bound The three positive numbers x, y, h have fixed sum 12. AM–GM gives

$$\frac{x + y + h}{3} \geq \sqrt[3]{xyh} \implies 4 \geq \sqrt[3]{V}.$$

Cubing yields $V \leq 64$.

Step 3: Equality Equality in AM–GM occurs exactly when

$$x = y = h = 4.$$

Thus the unique maximizing dimensions are $\boxed{x = 4, y = 4, h = 4}$ and

$$\boxed{V_{\max} = 64}.$$

Direct substitution satisfies the material constraint and confirms the volume.