

Question 8

Analyze

$$f(x, y) = \frac{xy}{1 + x^2 + y^2}.$$

Tasks

1. Prove strict bounds for its range.
2. Show every value between the bounds occurs.
3. Explain why the endpoints are approached but never attained.

Question 8 – Solution

Strategy Use $2|xy| \leq x^2 + y^2$ and then restrict to the lines $y = \pm x$.

Step 1: Bound Let $r^2 = x^2 + y^2$. Then

$$|f| \leq \frac{r^2/2}{1 + r^2} < \frac{1}{2}.$$

Thus the range is contained in $(-1/2, 1/2)$.

Step 2: Realize values On $y = x$,

$$f(x, x) = \frac{x^2}{1 + 2x^2},$$

which increases from 0 toward $1/2$. Solving $c = x^2/(1 + 2x^2)$ gives $x^2 = c/(1 - 2c)$ for every $0 < c < 1/2$. The line $y = -x$ similarly realizes every negative value, and $(0, 0)$ realizes 0.

Step 3: Conclusion Therefore

$$\boxed{\text{range}(f) = (-1/2, 1/2)}.$$

The denominator contains the extra positive 1, making equality in the bound impossible for finite inputs; the endpoints occur only as limits.